Quantum state Tomography

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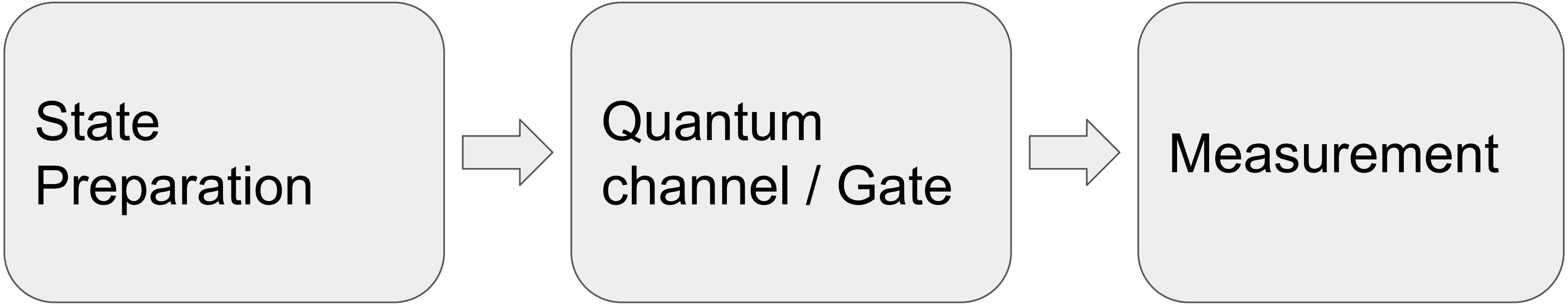
Problem statement



Density matrix

$$\rho \in \mathcal{L}(\mathcal{H}_2)$$
$$\rho = \rho *$$
$$tr[\rho] = 1$$
$$\rho \geq 0$$

Measurement operators

$$\sum_j \Pi_j = 1$$
$$p_j = tr[\Pi_j \rho]$$

Likelihood functional

$$LLH = \sum_j n_j \log(p_j)$$

Methods

- Hedged likelihood estimation
- Semidefinite programming
- Gradient descent
- Pseudo-inverse matrix

Preprocessing

- Make solution physical

$\rho_{est} = \rho_{est}^*$ (* means hermitian conjugation)
$\rho_{est} \geq 0$
$tr[\rho_{est}] \leq 1$

- Different matrix norms for error measurement

Hilbert-Schmidt

$$\Delta^{HS}(\rho', \rho) = \frac{1}{\sqrt{2}} \mathrm{Tr}[\,(\rho' - \rho)^2\,]^{1/2}$$

Trace-distance

$$\Delta^{T}(\rho', \rho) = \frac{1}{2} \mathrm{Tr}[\,|\rho' - \rho|\,]$$

Infidelity

$$\Delta^{IF}(\rho', \rho) = 1 - \mathrm{Tr}\Big[\sqrt{\sqrt{\rho'}\rho\sqrt{\rho'}}\,\Big]^2$$

Hedged likelihood estimation with iterative procedure

- Hedged likelihood functional

$$\mathcal{L}_H(\{n_j\};\rho) = (\det\rho)^\beta\,\mathcal{L}(\{n_j\};\rho)$$

where $\mathcal{L}(\{n_j\};\rho) = \prod_j p_j^{n_j}$ and probabilities $p_j = \mathrm{tr}\{\rho\,\Pi_j$

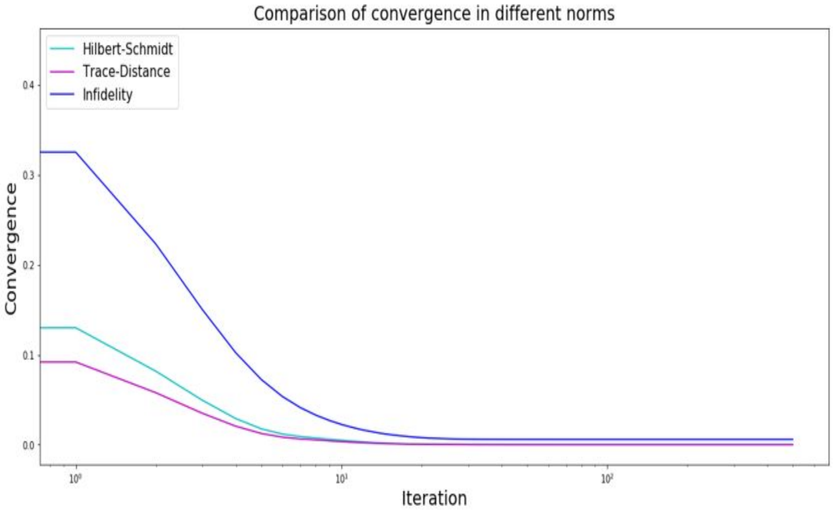
- HML iterative equations

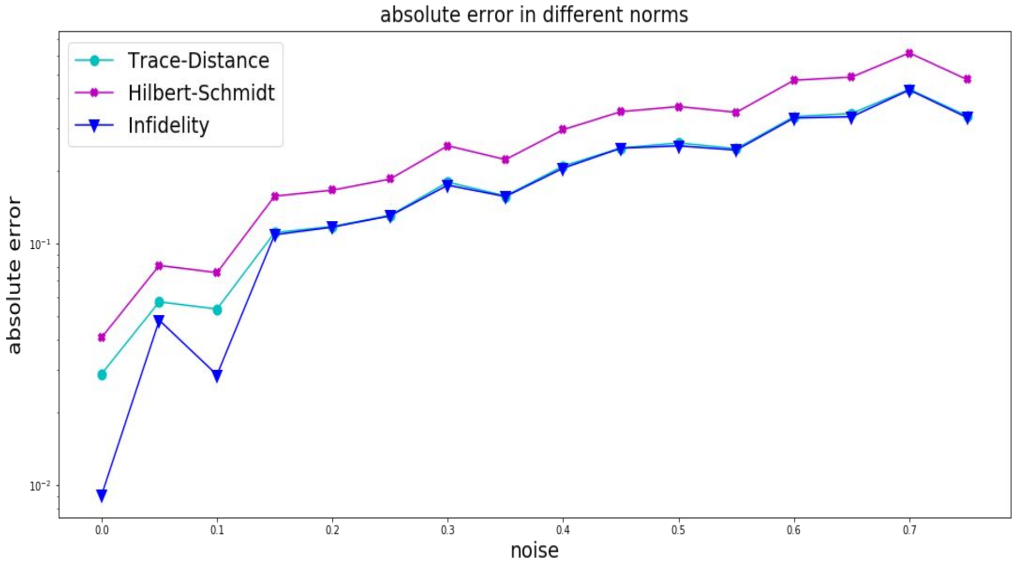
$$\rho_{k+1} = \frac{[\,1+\Delta_k\,]\rho_k[\,1+\Delta_k\,]}{\mathrm{tr}\{[\,1+\Delta_k\,]\rho_k[\,1+\Delta_k\,]\}},$$

$$\beta = \tfrac{1}{2}, \epsilon = \tfrac{1}{N}$$

$$\Delta_k = \frac{\epsilon}{2}[\,\beta(\rho_k^{-1} - D) + N(R_k - 1)\,]$$

Convergence and Absolute error for Likelihood method





Semidefinite Programming, Introduction

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- f_i convex, twice continuously differentiable

Lagrange Function

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) + \boldsymbol{\mu}^T (A\boldsymbol{x} - \boldsymbol{b}).$$

The Karush-Kuhn-Tucker conditions

$$\nabla_{\boldsymbol{x}} L(\hat{\boldsymbol{x}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \nabla f_0(\hat{\boldsymbol{x}}) + \sum_{i=1}^{m} \lambda_i \nabla f_i(\hat{\boldsymbol{x}}) + A^T \boldsymbol{\mu} = \mathbf{0},$$

$$\lambda_i \ge 0, \;\; \lambda_i f_i(\hat{\boldsymbol{x}}) = 0, \;\; i = \overline{1, m};$$

$$f_i(\hat{\boldsymbol{x}}) \le 0, \;\; A\hat{\boldsymbol{x}} = \boldsymbol{b}.$$

$$B(x, \mu) = f(x) - \lambda \sum_i ln(f_i(x))$$

Barrier function

$$p_B = x + \frac{1}{\lambda} X^2 (A^* \mu - b)$$

Newton direction

Semidefinite Programming

We have 2 estimators:

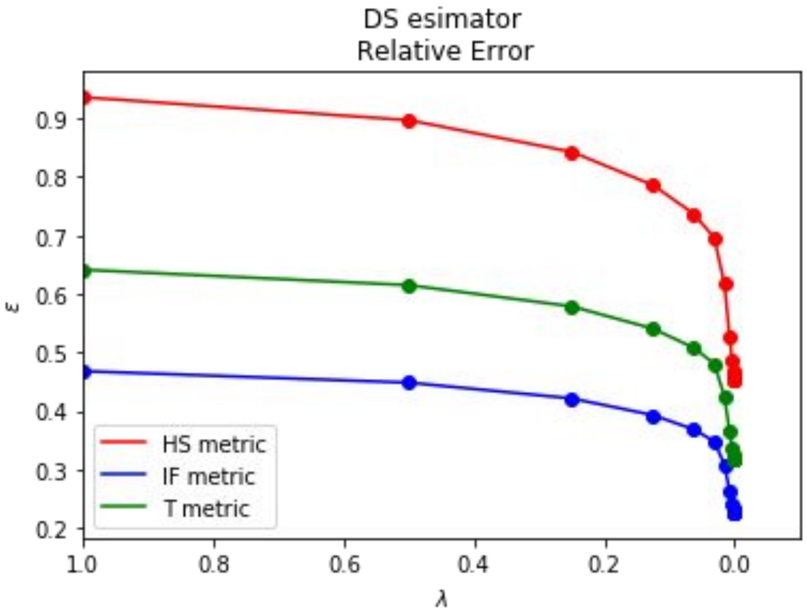
The matrix Dantzig selector

$$\hat{\rho}_{DS} = arg\min_X \|X\|_{tr}, \text{ s.t. } \|A^*(A(X) - y)\| < \lambda$$

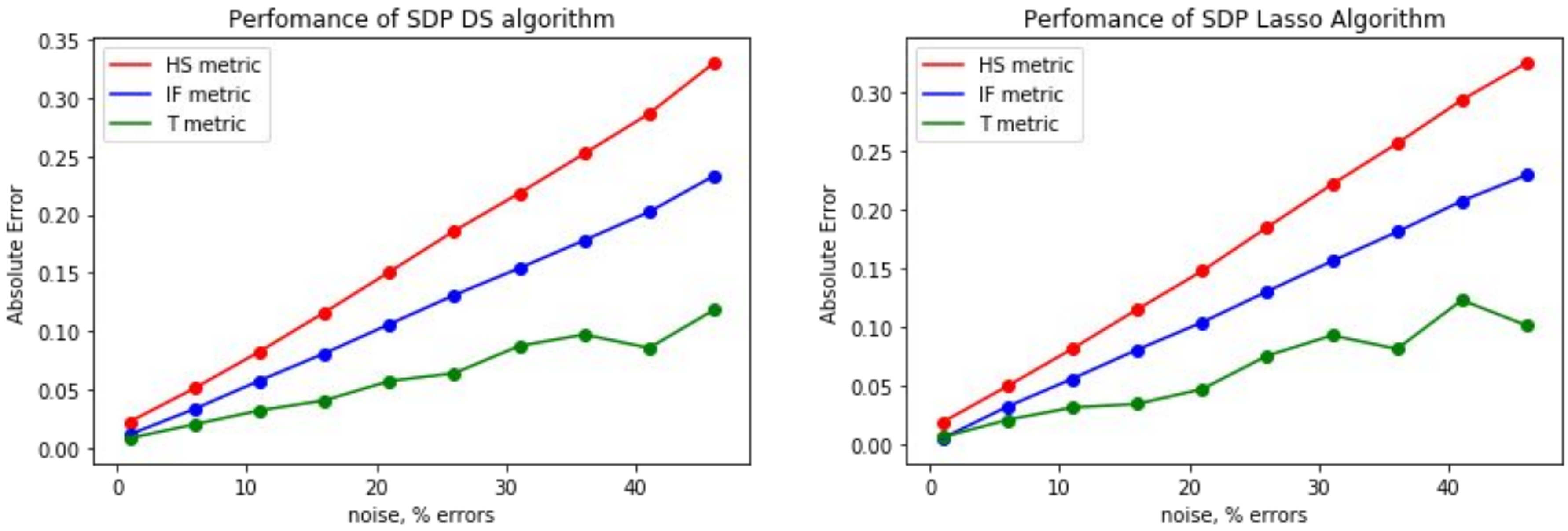
The Lasso matrix:

$$\hat{\rho}_{Lasso} = arg\min_X \frac{1}{2}\|A(X) - y\|_2^2 + \mu\|X\|_{tr}$$

$$\|X\|_{\text{tr}} = \text{Tr}|X|, \text{ where } |X| = \sqrt{X^\dagger X}.$$



Semidefinite Programming



Direct Gradient Algorithm

Maximize likelihood functional

$$\delta \log \mathcal{L}(\{n_j\};\rho) = \sum_j f_j \frac{\delta p_j}{p_j} = \mathrm{tr}\{R\delta\rho\}$$

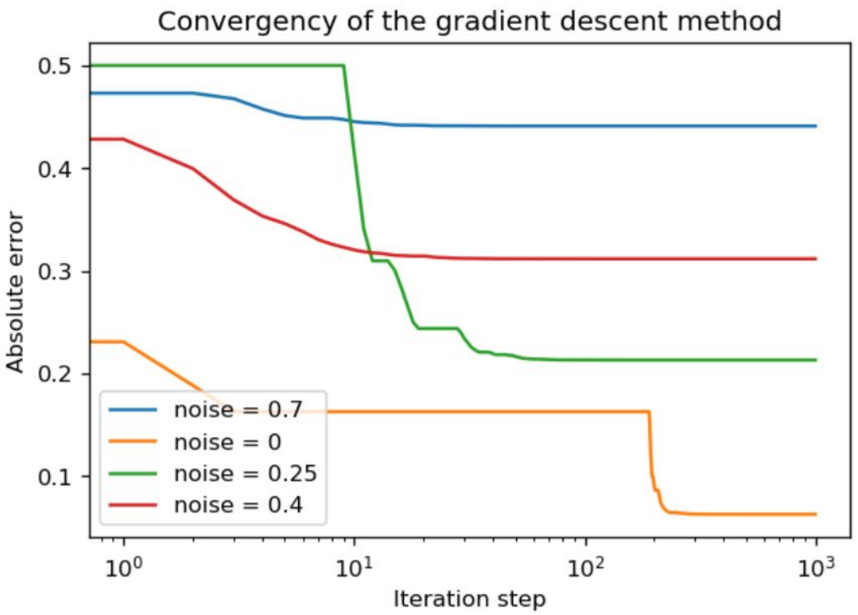
where

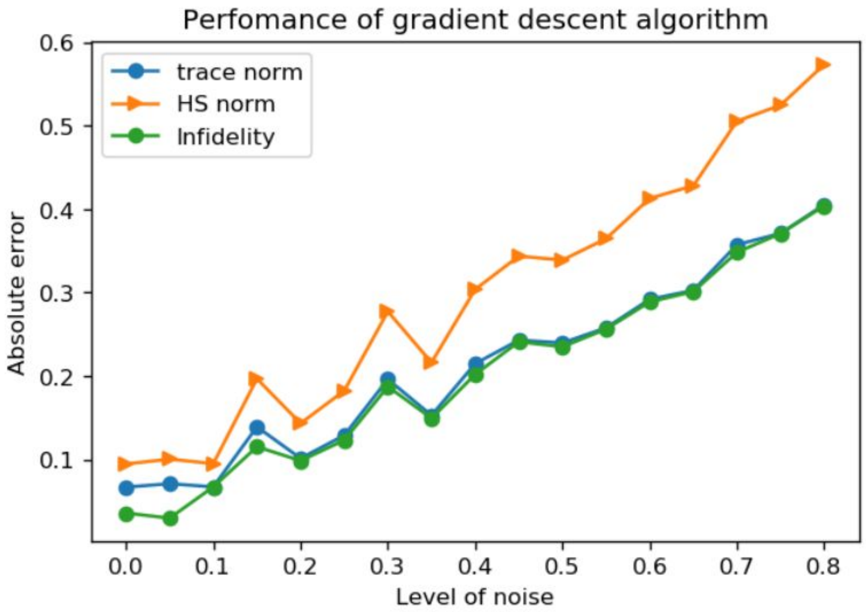
$$R = \sum_j \frac{f_j}{p_j}\Pi_j$$

Iterative procedure:

$$\rho_{k+1} = \frac{\left[1+\frac{\epsilon}{2}\left(R_k-1\right)\right]\rho_k\left[1+\frac{\epsilon}{2}\left(R_k-1\right)\right]}{\mathrm{tr}\left\{\left[1+\frac{\epsilon}{2}\left(R_k-1\right)\right]\rho_k\left[1+\frac{\epsilon}{2}\left(R_k-1\right)\right]\right\}},$$

Direct Gradient Algorithm





Pseudo-Inverse Matrix Method

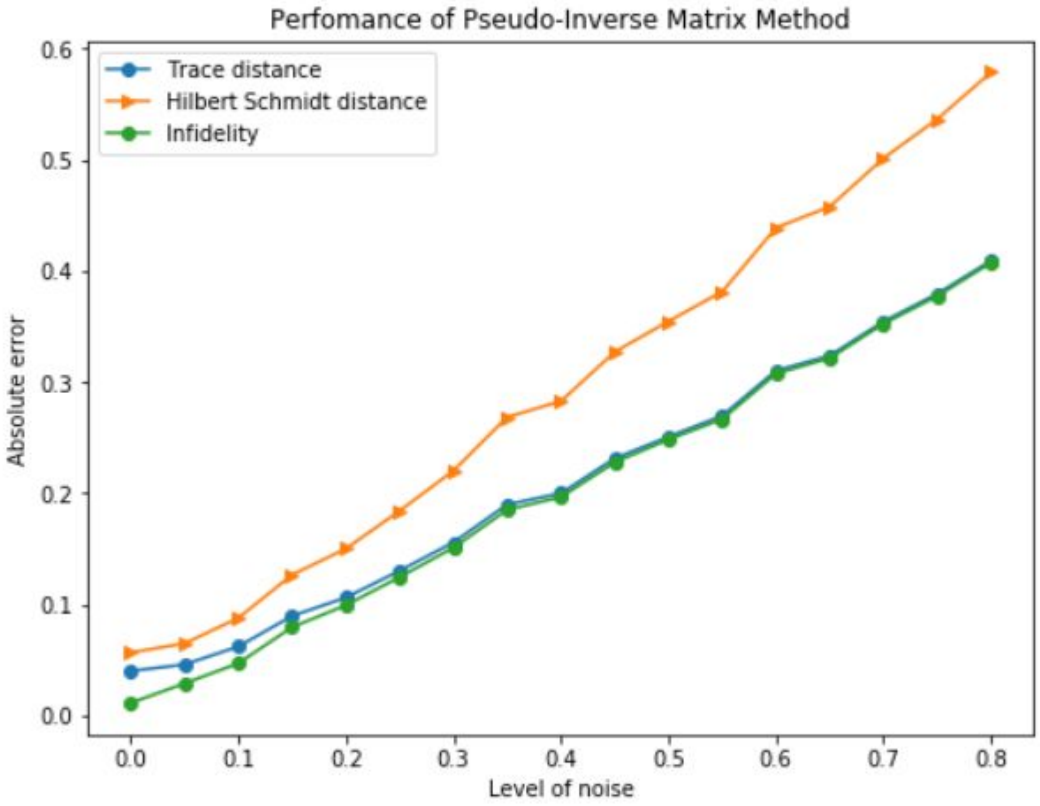
Density matrix and Measurement operator parametrization via Pauli matrices

$$\rho = \frac{I + \sum_k (a_k \cdot \sigma_k)}{2}$$

$$E_j = s_0 I + \sum_k s_{k,j} \sigma_k$$

$$p_j^{(i)} = (\mathbf{s_j^{(i)}})^{\mathbf{T}} \mathbf{a}$$

$$\mathbf{p} = \mathbf{S}\mathbf{a}$$



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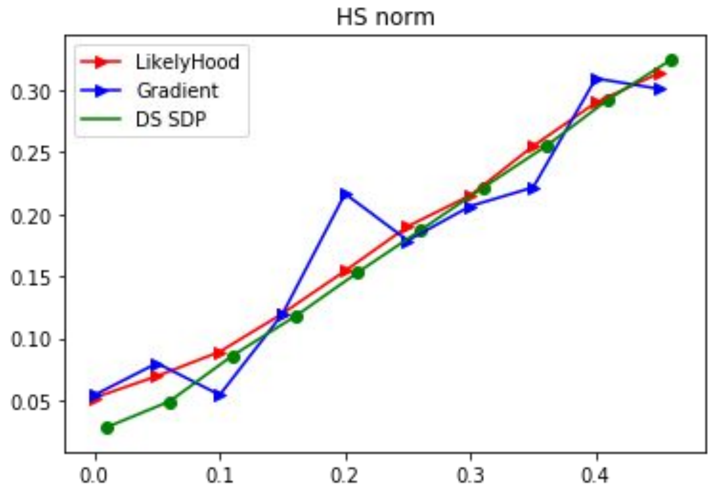
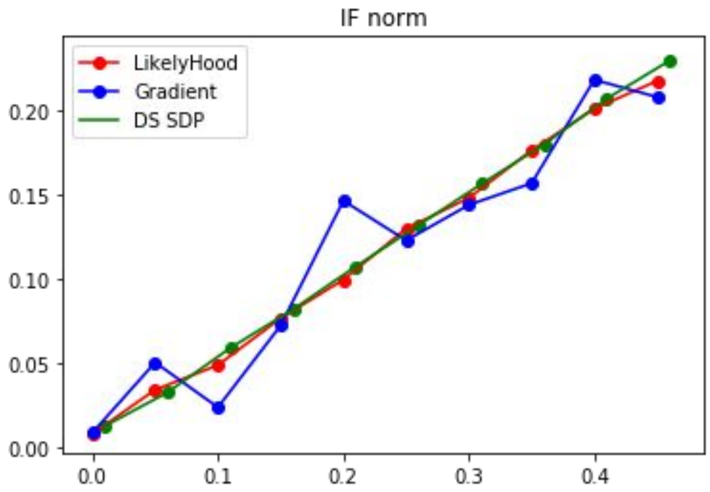


noise



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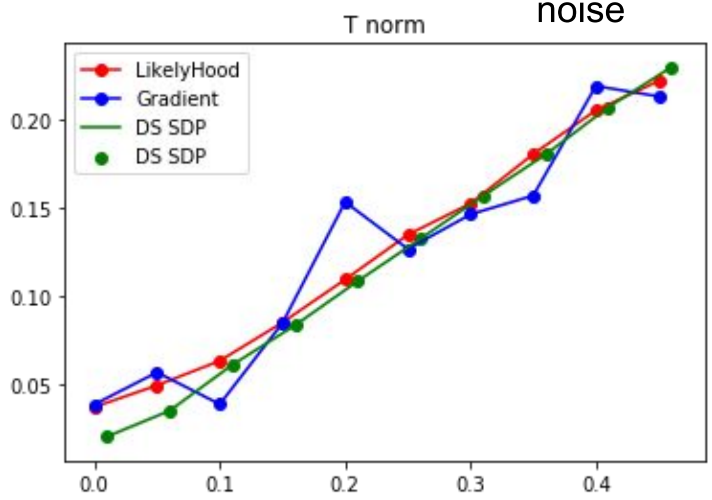

noise

Conclusion

Pseudo-Inverse Matrix

+ easy to implement
- unphysical solution

Direct Gradient

- slow convergence
- unphysical solution

SDP method:

+ fast convergence
+ physical solution

Hedged Likelihood:

+ fast convergence
+ solve zero-probabilities problem
- unphysical solution

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