Enhanced image approximation using shifted rank-1 reconstruction

Team 20

Gleb Balitskiy, Andrei Dzis, Nikolay Kozyrskiy, Nikolay Skuratov

Introduction

Low rank approximation of a data matrix $A \in \mathbb{C}^{M \times N}$ is of great interest in many applications. For example:

1. video denoising
2. object detection
3. subspace segmentation
4. seismic data processing
5. …

Problem Statement

The problem can be formulated as follows: Given the matrix A and a small number L one seeks for vectors $u^1, \cdots, u^L \in \mathbb{C}^M, v^1, \cdots, v^L \in \mathbb{C}^N$ and coefficients $\sigma_1, \cdots, \sigma_L$ such that

$$\left\| A - \sum_{k=1}^{L} \sigma_k u^k \left(v^k\right)^* \right\|_F \text{ is small.}$$

Problem statement

We seek to find an approximation of A using L rank-1 matrices $u^k\left(v^k\right)^*$ where the columns of each matrix can be shifted arbitrarily. Let $\lambda^k \in \mathbb{Z}^N$ and $S_{\lambda^k} : \mathbb{C}^{M\times N} \to \mathbb{C}^{M\times N}$ be the operator that circularly shifts the j-th column of a matrix by λ^k_j.
Thus, the following problem can be proposed in terms of optimization problem:

$$\min_{\substack{\lambda^1,\ldots,\lambda^L \\ u^1,\ldots,u^L \\ v^1,\ldots,v^L}} \left\| A - \sum_{k=1}^{L} S_{\lambda^k}\left(u^k\left(v^k\right)^*\right)\right\|_F,$$

where $S_{\lambda^k}\left(u^k\left(v^k\right)^*\right)$ a shifted rank-1 matrix.

Methodology

Definition 1 (Shift operators): Denote the columns of the matrix A by $A = [a^1, \ldots, a^N]$. For $\lambda \in \mathbb{Z}^N$ define the shift operator $S_\lambda : \mathbb{C}^{M\times N} \to \mathbb{C}^{M\times N}$ as

$$S_\lambda A := [\tilde{S}^{\lambda_1} a^1, \ldots, \tilde{S}^{\lambda_N} a^N]$$

Theorem 1: For arbitrary $\lambda \in \mathbb{Z}^N$ the operator S_λ is linear. The inverse operators are given by $S_\lambda^{-1} = S_{-\lambda}$ Furthermore $||S_\lambda A||_F = ||A||_F$ holds. Given $\lambda' \in \mathbb{Z}^N$ we obtain $S_\lambda S_{\lambda'} = S_{\lambda+\lambda'}$.

Definition 2 (Shifted rank-1 matrices): We call $A \in C^{M\times N}$ a shifted rank-1 matrix, if $A = S_\lambda(u, v*)$ for some vectors $u \in \mathbb{C}^M, v \in \mathbb{C}^N and \lambda \in \mathbb{Z}^N$

Methodology

Having a closer look at iteration step:

$$\min_{\lambda,u,v} ||A - S_\lambda(uv^*)||_F \rightarrow \min_{\lambda,u,v} ||S_{-\lambda}(A) - uv^*||_F$$

Let $\sigma_1, \ldots, \sigma_K$ be the singular values of $S_{-\lambda}(A)$, then:

$$||S_{-\lambda}(A) - uv^*||_F^2 = \sum_{k=2}^{K} \sigma_k^2 = ||S_{-\lambda}(A)||_F^2 - \sigma_1^2 = ||A||_F^2 - ||S_{-\lambda}(A)||_2^2$$

Thus, we can find λ by solving:

$$\max_{\lambda \in \mathbb{Z}^N} ||S_{-\lambda}(A)||_2^2 \tag{1}$$

So the shifted rank-1 approximation can be summarized as follows

Methodology

Algorithm 1

1: Input: A, L
2: for $k = 1, \ldots, L$ do
3: Solve (1) for λ^k
4: Calculate (1) for λ^k using SVD of $S_{-\lambda}(A)$
5: Update $A \leftarrow A - S_\lambda\left(u^k\left(v^k\right)^*\right)$
6: end for
7: return λ^k, u^k, v^k for $k = 1, \ldots, L$

The fourth step of this algorithm is done by algorithm 3, using routines from algorithm 2.

Methodology

Algorithm 2

```
Input: $A$
Set $s = 1$
while $s \neq 0$ do
    Calculate $\hat{u}$ the first left singular vector of $\hat{A}$
    Set $B = \left| F^{-1}(\bar{\hat{u}} \odot \hat{A}) \right|^2$
    Subtract the first row of $B$ from every row
    Get the position $(k, s)$ of the maximum in $B$
    Shift the $k$ -th column of $A$ by $-s$
    Update $\lambda_k \leftarrow \lambda_k + s$
end while
return $\lambda$
```

Methodology

Algorithm 3

1: Input: A

2: Calculate $B = \left|F^{-1}(|\hat{A}| \odot \hat{A})\right|^2$

3: Set λ to the indices of the maximum values of B

in each column and update $A \leftarrow S_{-\lambda}A$

4: Update λ and A using Algorithm 2 with $|\hat{u}| = |\hat{u}^{opt}| \odot \text{phase}(\hat{u})$

5: Update λ and A using Algorithm 2

6: return λ

Obtained results

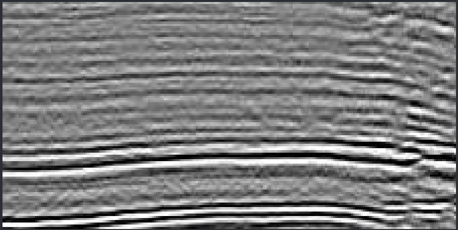


Figure: Original

Obtained results

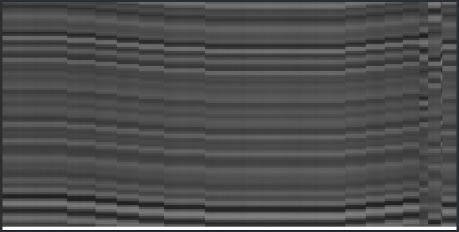
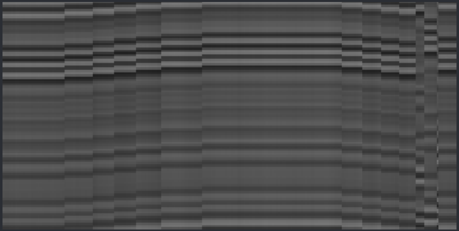


Figure:



Brief discussion of the results

- Shift operators and shifted rank-1 matrices were introduced.
- With their help we found a shifted rank-1 approximation of our input.
- Proposed approximation showing itself slightly worse than other methods.
- Parameters, obtained during the work of the algorithm, demonstrated, that they are very informative and could be used in extraction of crucial information.

Thank you for attention!