Basics of NLA

NLA course

2019

If you do not know answers to the following questions or can not solve the following problems, you will likely go to re-examination.

1. Questions

1. Floating vs. fixed point representation of numbers.
2. Definitions of vector and matrix norms. Basic norms: p-norms (vector and matrix), Frobenius norm.
3. Complexity of basic linear algebra operations: e.g. matrix-vector, matrix-matrix products.
4. Definition of eigenvalues and eigenvectors of a matrix. Characteristic equation.
5. What is singular value decomposition?
6. Definition of Hermitian and unitary (symmetric, orthogonal) matrices. Properties of their eigenvalues.
7. Definition of symmetric positive definite matrix.
8. Definition of normal matrices and their properties.
9. Definition, existence and uniqueness of basic matrix decompositions: LU, QR, Cholesky, Schur, SVD, skeleton, eigendecomposition. Computational complexity of these decompositions.
10. Definition of condition number.
11. Definitions of key matrices: Fourier, permutation, Householder, Givens, Hessenberg, triangular, Toeplitz, circulant.
12. Formulation of Eckart-Young theorem.
13. Formulation of the QR algorithm.
14. Least-squares problem. Definition of a pseudoinverse.
15. Power method and how it converges.
16. CSR format for sparse matrix storing.
17. Richardson iteration. Optimal parameter in the case of symmetric positive definite matrix.
18. Krylov subspace. Idea of main Krylov methods: CG, MINRES, GMRES, bicgstab. Differences between these methods and when to apply them.
19. Idea of ILU preconditioning.
20. Definition of Toeplitz matrix
21. Fast Fourier transform and how it helps to multiply vactor by Toeplitz matrix fast.

2. Problems

1. Find the determinant of the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

2. Prove that the matrix $xy^\top$ has rank equals to one.

3. Find at least one eigenpair of matrix $A = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. How many eigenvalues are there?

4. Find pseudo-inverse of a scalar $(c)^\dagger$?

5. Find pseudo-inverse of a vector $(a)^\dagger$?

6. Find pseudo-inverse of a rank one matrix $\left(ab^\top\right)^\dagger$?

7. Find singular values of matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

8. Find any eigenvector of the following matrix:

$$\begin{bmatrix} 5 & 0 & 5 \\ -1 & 0 & 1 \end{bmatrix}$$

9. Write down formula for Householder reflection matrix

10. Write down formula for Givens rotation matrix

11. Prove that $\|xy^\top\|_F = \|xy^\top\|_2 = \|x\|_2\|y\|_2$ for any $x, y \in \mathbb{C}^n$

12. Solve $\frac{\partial}{\partial a} x^\top a$

13. Solve $\frac{\partial}{\partial A} \mathrm{tr}(A^\top B)$

14. Find the gradient $\nabla f(x)$ and hessian $f''(x)$, if $f(x) = \dfrac{1}{2}\|Ax - b\|_2^2$.

15. Show that if matrix is triangular and unitary, then it is diagonal