Midterm preparation problems

November 15, 2019

1. What is the name of the following quotient:

$$\mathcal{R}_A(x) = \frac{(Ax, x)}{(x, x)}?$$

   (a) Reilly (b) Raleigh (c) Rayleigh (d) O RLY?

2. What is the LU-decomposition of a matrix? Does it always exist for square matrices? If not, give a criteria. Is it unique?

3. Formulate the QR algorithm. Where does it converge? What tricks can be used to avoid $\mathcal{O}(n^4)$ complexity?

4. What is a normal matrix? What can we say about eigenvectors of normal matrices?

5. What is the skeleton decomposition? How many unique parameters define the skeleton decomposition?

6. Find the distance between singular matrix and the closest nonsingular in $\|\cdot\|_2$ and $\|\cdot\|_\infty$ norms.

7. Find pseudoinverse of a matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$

8. Find $\text{cond}_2 \begin{bmatrix} 0 & \epsilon \\ 1 & 0 \end{bmatrix}$.

9. Show that strictly diagonally dominant matrices are nonsingular.

10. Let matrix $A \in \mathbb{C}^{n\times n}$ be given by its skeleton decomposition $A = BC^*$, where $B, C \in \mathbb{C}^{n\times r}$. Suggest an algorithm that finds SVD of $A = U_r\Sigma_r V_r^*$ in $\mathcal{O}(nr^2 + r^3)$ operations assuming that B and C are given.

11. Let $u, v \in \mathbb{R}^{n\times 1}$. Find $\det(I + uv^*)$.

12. Find $\|F_n\|_\infty$ and $\|F_n\|_2$, where F_n is the Fourier matrix.

13. What is the QR-decomposition of a matrix? Does it always exist? Is it unique?

14. What is the Schur decomposition? Let matrix be symmetric. What is its Schur decomposition?

15. What is the Cholesky factorization? What is the criteria of its existence? Does it require pivoting for stability?

16. Let $A \in \mathbb{C}^{m\times n}$. Prove that $\|A\|_1 = \max\limits_{1\le j\le n} \sum\limits_{i=1}^{m} |a_{ij}|$

17. Let $\mu \in \lambda(A + E)$, $\mu \notin \lambda(A)$. Prove that (Bauer-Fike theorem)

$$\frac{1}{\|(A - \mu I)^{-1}\|_2} \le \|E\|_2.$$

18. Let matrix $A \in \mathbb{C}^{n\times n}$ be given by its skeleton decomposition $A = BC^*$, where $B, C \in \mathbb{C}^{n\times r}$. Suggest an algorithm that finds SVD of $A = U_r\Sigma_r V_r^*$ in $\mathcal{O}(nr^2 + r^3)$ operations assuming that B and C are given.

19. Find the second singular value $\sigma_2(A)$ of matrix $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

20. Give definition of the singular value decomposition (SVD). Find an SVD of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Is it unique?

21. Give definition of the QR decomposition. Does the QR decomposition of the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ exist?

22. What matrix is called diagonalizable? Is Householder matrix diagonalizable? Why?

23. Formulate the QR algorithm. Does the matrix produced by the QR algorithm applied to $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ converge to an upper triangular matrix? Why?

24. Let $A \in \mathbb{C}^{m\times n}$. Prove that $\|A\|_1 = \max\limits_{1\le i\le n} \sum\limits_{i=1}^{m} |a_{ij}|$.

25. Let $U \in \mathbb{C}^{n\times k}, k < n$ be a matrix so that $U^*U = I_k$. Find a pseudoinverse of UU^*.

26. Is it true that if $(Ax, x) \ge 0$ for all $x \in \mathbb{R}^n$, then A is Hermitian? Prove or provide a counter example.

27. Let $u, v \in \mathbb{R}^{n\times 1}$. Find $\det(D + uv^\top)$, where $D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$ and $\det(D) \ne 0$.

28. Give definition of the skeleton decomposition. Find a skeleton decomposition of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

29. Give definition of the LU decomposition. Does the LU decomposition of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix}$ exist? Why?

30. What matrix is called diagonalizable? Is Fourier matrix diagonalizable? Why?

31. Formulate the QR algorithm. What is the convergence rate of the QR algorithm applied to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$?

32. Decomposition

$$A = H + iK, \quad H = H^*, \quad K = K^*, \quad i^2 = -1$$

is called the Hermitian decomposition of A. Does it always exist?
Prove that if $(Ax, x) \ge 0$ for all $x \in \mathbb{C}^n$, then A is Hermitian.

33. Give definition of the QR decomposition. Does the QR decomposition of the matrix $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ exist? Why?

34. Formulate the QR algorithm. Does the matrix produced by the QR algorithm applied to $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ converge to an upper triangular matrix? Why?

35. Compute SVD of the matrix

$$A = \begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{5} \\ \vdots \\ \sqrt{2n-1} \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 & \dots & (-1)^n \end{bmatrix}$$

36. Find the Givens rotation matrix G, which reflects vector $x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ onto vector $e = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, such that $Gx = \|x\|_2 e$.